Results at a glance. My research extends representation stability for FI-modules to stable ∞-category theory and shows that representation stability can be understood as a facet of a new functor calculus for FI analogous to Michael Weiss' orthogonal calculus introduced in [Wei95]. I define a Taylor tower and Taylor coefficients for FI-objects and describe structure maps between these coefficients. I prove, in analogy to a result of Arone and Ching in [AC14], that a Taylor tower is determined by its Taylor coefficients and these structure maps up to the vanishing of certain Tate constructions. Over a field of characteristic 0, I give an explicit calculation relating an FI-module and its coefficients. In a forthcoming generalization of this work, I show that any ∞-category admitting a Cartesian fibration to FI hosts a similar functor calculus.

The ∞-categorical perspective on representation stability broadens the subject in several new directions. First, it reveals a larger class of well-behaved FI-modules, not strictly representation stable, which deserve consideration in the context of representation stability. Almost all examples of representation stable FI-modules of interest arise as the cohomology of some family of objects indexed by $\mathsf{FI}^{\mathrm{op}}$ and hence naturally carry extension data. The Taylor coefficients compactly encode the structure of a representation stable FI-module as well as this extension data. FI-calculus also sheds light on the behavior of representation stable FI-modules in the "pre-stable" range. More broadly, FI-calculus allows for the consideration representation stability in tandem with extraordinary cohomology theories.

Finally, FI-calculus opens the door to myriad future projects and further extensions. The most important of these is the fact that the most fundamental techniques of FI-calculus extend to a wide range of domain (∞-)categories other than FI allowing for the ideas of functor calculus to be applied in a host of new settings. Other extensions involve establishing theorems relating FI-calculus and Goodwillie calculus and exploring the possible existence of other cousins of orthogonal calculus with a longer-term view toward developing a general framework formally unifying FI-calculus and orthogonal calculus.

Background on ∞-category theory. My research lies within the field of homotopy theory, with an especial focus on problems with an ∞-categorical flavor. Homotopy theory is the branch of mathematics concerned with structures arising from and applications of invariants which do not distinguish between "weakly equivalent" objects of some species – classically, topological spaces, as homotopy theory emerged from the field of algebraic topology. It is almost always desirable that structure-preserving functions between objects of interest induce corresponding functions between invariants – that is to say, in the language of category theory, that the invariants be functorial – and it was to formalize this behavior that category theory was introduced.

Because the weak equivalences which appear in homotopy theory are not in general isomorphisms in the 1-categorical sense but rather a proper generalization thereof, it can be useful to keep track of data relating the composite of a pair of "weakly inverse" morphisms to the respective identity morphisms – witnessing the fact that the pair were "weakly inverse" – and this leads directly to the eponymous homotopies between morphisms, to be conceptualized as "paths" connecting two morphisms. In fact, there arise further homotopies between homotopies and so on, all of which it is fruitful to keep track of. Grothendieck's celebrated homotopy hypothesis postulates that the correct structure for recording this data is a space, whether in the guise of a topological space, a simplicial set, or some other model, and categories in which the collections of parallel morphism carry this spatial structure are called ∞-categories in reference to the existence of homotopies between homotopies and so on.

The first technology for working with categories equipped with this homotopical structure was Quillen's model categories, introduced in [Qui67], and an extremely robust theory thereof was developed in books such as [Hir09], [Hov07], and others. Beginning in the 1970s with Boardman and Vogt's "weak Kan complexes", introduced in [BV73], and especially since the beginning of the twenty-first century with, for example, [Rez00] and [Ber07], other technologies for working with ∞-categories have been developed and compared. These definitions have a reputation for abstruseness, but they have the payoff of significantly streamlining many proofs, and – once the foundations have been dealt with – largely succeed in restoring to ∞-category theory the sleekness enjoyed by many 1-category-theoretic arguments, and a great deal of ∞-category theory has been fleshed out in works such as [Lur09], [Lur17], [Lur18], [RV22], [GH15], and others.

Background on representation stability and functor calculus. My dissertation research introduces a new flavor of functor calculus extending representation stability to stable ∞-category theory[1].

Let FI be the category of finite sets and injections and $\mathbb{Q}\mathbf{Vect}$ the category of rational vector spaces. A functor $\mathsf{FI} \to \mathbb{Q}\mathbf{Vect}$, called an FI-module, determines a sequence of representations of the symmetric groups $\mathfrak{S}_n$. Representation stability is a phenomenon enjoyed by many FI-modules of interest – especially including the cohomology of many moduli spaces and configuration spaces – which ensures that the representations determined by the FI-module eventually follow a certain pattern. The theory has its origins in [CF13], was articulated in the language of FI-modules in [CEF15], and in [Chu+14] the authors show that over Noetherian rings, an FI-module is representation stable and objectwise finite-dimensional if and only if it is finitely generated.

On the other hand, functor calculus refers to a family of techniques within homotopy theory concerned with approximating functors between certain (∞-)categories by other, more well-behaved "polynomial" or "excisive" functors. These approximations form a tower (or some generalization thereof) analogous to the Postnikov tower in the homotopy theory of spaces, and the fibers of each stage of the tower, analogous to the role of Eilenberg-MacLane spaces, are described by "coefficient objects" – often spectra equipped with an action of some group.

The most prominent member of this family of calculi is Goodwillie calculus, originally developed by Tom Goodwillie in [Goo90], [Goo91], and [Goo03]. Today, Goodwillie calculus has developed into a rich subfield of homotopy theory with an array of results and applications too numerous to list here.

Other flavors of functor calculus include orthogonal calculus, introduced by Michael Weiss in [Wei95] and dealing with functors from the category of Euclidean spaces to topological spaces; embedding calculus, developed by Tom Goodwillie and Michael Weiss and introduced in [Wei96] and dealing with space-valued presheaves

[1] Beware that the term "stable" regrettably has two distinct meanings here.

on categories of manifolds and embeddings; and several others. It is to orthogonal calculus that my work is most reminiscent.

FI-calculus. Fix $\mathcal{V}$ a stable, presentable ∞-category. A stable ∞-category is the ∞-categorical analog of an abelian category, and presentability is a (co)completeness condition together with a set-theoretic tameness condition. I define a standard n-cube to be a diagram in FI, determined by a pair of sets $S \subseteq S'$ such that $|S' \setminus S| = n$ and consisting of all intermediate sets $S \subseteq T \subseteq S'$ along with the inclusion morphisms. I call a functor $\mathsf{FI} \to \mathcal{V}$ an "FI-object" and denote the ∞-category of FI-objects $\mathsf{FI}\mathcal{V}$. I define an *n-polynomial* FI-object to be one sending all standard $n+1$-cubes to limit diagrams (also called "Cartesian cubes") and denote the ∞-category of n-polynomial FI-objects $\mathrm{Poly}_n\mathcal{V}$. I say that an FI-object E is *polynomial* if there exists some $n \in \mathbb{N}$ such that $E \in \mathrm{Poly}_n\mathcal{V}$. I show, in analogy with a theorem from representation stability, that an FI-object is n-polynomial if and only if it is left Kan extended from $\mathsf{FI}_{\leq n}$, the full subcategory of FI spanned by sets of cardinality at most n.

As in other flavors of functor calculus, there is a "Taylor tower" of universal n-polynomial approximations $\mathbf{P}_nE$ under a given FI-object E. I call an FI-object E *n-homogeneous* if E is n-polynomial and $\mathbf{P}_{n-1}E \cong 0$ and prove an equivalence

$$\mathrm{Hmg}_n\mathcal{V} \simeq \mathfrak{S}_n\mathcal{V}$$

between the ∞-categories of n-homogeneous FI-objects and of $\mathfrak{S}_n$-objects in $\mathcal{V}$ – a result with direct analogs in orthogonal calculus and in Goodwillie calculus. The layers of the Taylor tower of a given $E \in \mathsf{FI}\mathcal{V}$, the FI-objects

$$\mathbf{D}_nE \overset{\text{def}}{=} \mathrm{fib}\,\mathbf{P}_nE \to \mathbf{P}_{n-1}E$$

are n-homogeneous and hence determine $\mathfrak{S}_n$-objects, which I call the *Taylor coefficients* $\mathbf{C}_nE$ of E.

Surprisingly, there exist maps between the Taylor coefficients of an FI-object making those coefficients – a priori only a symmetric sequence – into an FI-object themselves. The first main question of my dissertation is to establish how much information can be recovered from these Taylor coefficients along with their FI-object structure. To address this question in full generality, I introduce the ∞-category

$$\mathrm{FTT}\mathcal{V} \overset{\text{def}}{=} \lim \cdots \xrightarrow{\mathbf{P}_n} \mathrm{Poly}_n\mathcal{V} \xrightarrow{\mathbf{P}_{n-1}} \cdots \xrightarrow{\mathbf{P}_0} \mathrm{Poly}_0\mathcal{V}$$

of "formal Taylor towers." $\mathrm{FTT}\mathcal{V}$ is the natural domain of the aggregate Taylor coefficient functor $\mathbf{C}$, and under good conditions (e.g. when $\mathcal{V}$ is $\mathbb{Q}$-linear) I prove that $\mathbf{C}$ determines an equivalence of ∞-categories

$$\mathbf{C} : \mathrm{FTT}\mathcal{V} \simeq \mathsf{FI}\mathcal{V}$$

My second main result deals with the specialization to the case $\mathcal{V} = \mathcal{S}p^{\mathbb{Q}}$ of functors from FI to the ∞-category of rational chain complexes and establishes FI-calculus as a direct generalization of representation stability to the setting of stable ∞-categories. I show that if an FI-chain complex E is n-polynomial for some $n \in \mathbb{N}$, then its homology is representation stable; that if an FI-module $E : \mathsf{FI} \to \mathbb{Q}\mathbf{Vect}$ is representation stable, then there exists $n \in \mathbb{N}$ such that, when E is considered as a discrete FI-chain complex, E agrees with $\mathbf{P}_nE$ outside of a finite range; and finally that the $\mathfrak{S}_n$-representations appearing in the stable range of the homology of an n-polynomial FI-chain complex E can be directly read off from the homology of the coefficient FI-chain complex $\mathbf{C}E$.

Work in progress and future directions. One of the several major benefits of the ∞-categorical perspective on representation stability is the wide range of new avenues it opens up for further research. More specifically, the techniques used in FI-calculus give rise to several families of generalized functor calculus. In each of these, there is a classification of homogeneous functors by certain Taylor coefficients. In general, it is an interesting question to investigate what morphisms exist between these coefficients or what other structure may tie them together and under what conditions these structures allow for the recovery of a Taylor tower.

One generalization begins with the observation that FI is the category of 0-dimensional manifolds and embeddings. The fundamental framework of FI-calculus can be extended to Embd_d, the ∞-category of d-dimensional manifolds and embeddings. In this case the standard cubes are (the opposites of) the cubes relevant to the Goodwillie-Weiss embedding calculus of [Wei96], and the same techniques which apply for FI-calclus allow for a classification of n-homogeneous functors $\mathrm{Embd}_d \to \mathcal{V}$ by coefficient functors with domain the full sub-$infty$-category of Embd_d spanned by the disjoint union of n open d-disks.

It is important to note that this does not reproduce the Goodwillie-Weiss embedding calculus: embedding calculus is concerned with functors with domain $\mathrm{Embd}_d^{\mathrm{op}}$ and yields a markedly different theory in which polynomial functors are obtained as right Kan extensions from full sub-∞-categories of disks. The extension of FI-calculus to higher dimensional manifolds instead has more interaction with factorization homology as developed by Ayala, Francis, and others, but would represent a novel approach to studying functors from the category Embd_n.

Generalizing along a complementary direction, FI-calculus is the terminal example in a family of functor calculi. Given any Cartesian fibration

$$\varpi : \mathcal{D} \to \mathsf{FI}$$

one may "lift" FI-calculus along ϖ to obtain a functor calculus for functors $\mathcal{D} \to \mathcal{V}$. In this case, Taylor coefficients are functors

$$\mathbf{C}_n E : \varpi^{-1}(\mathfrak{S}_n) \to \mathcal{V}$$

Examples include ∞-categories of totally ordered finite sets, cyclically ordered finite sets, and directed or undirected graphs, as well as more involved examples: e.g. given a manifold M, the ∞-category M**Braid** with objects finite sets of distinct marked points in M and morphisms given by braids from one set of points to another. When $M = \mathbb{R}^2$, this recovers the category of braids, a category also arising in the homological stability framework of Oscar Randal-Williams and Nathalie Wahl in [RW17].

When ϖ is a right fibration, the Taylor coefficients of a functor or a formal Taylor tower, as when $\mathcal{D} = \mathsf{FI}$, assemble into a functor $\mathcal{D} \to \mathcal{V}$. Subject to appropriate finiteness conditions, it is sensible to ask if the vanishing of certain Tate constructions, as in the case of FI, permit the reconstruction of a Taylor tower from its aggregate Taylor coefficient functor. This is work which presently engages me. It is also of interest in this setting to investigate whether these functor calculi categorify representation stability phenomena.

If ϖ has a braided monoidal structure compatible with its Cartesian structure, we obtain a refined notion of polynomial functor (and hence also homogeneous functor) $\mathcal{D} \to \mathcal{V}$ indexed by a poset whose elements are built from sets of objects in $\mathcal{D}$.

The preceding generalizations can be also combined: a Cartesian fibration

$$\varpi : \mathcal{D} \to \mathrm{Embd}_d$$

yields a functor calculus for functors $\mathcal{D} \to \mathcal{V}$ including a classification theorem for homogeneous functors.

The ∞-categorical perspective on representation stability also invites investigation of the interactions between representation stability and Goodwillie calculus. In [BE16], David Barnes and Rosona Eldred investigate the interaction between orthogonal calculus and Goodwillie calculus by composing functors of interest in Goodwillie calculus with the functor

$$V \mapsto S^V : \mathcal{J} \to \mathcal{S}$$

sending a vector space to its one-point compactification, where $\mathcal{J}$ is the ∞-category of Euclidean spaces and $\mathcal{S}$ is the ∞-category of topological spaces. I am interested in similar questions: given stable presentable ∞-categories $\mathcal{V}$ and $\mathcal{W}$ and functors

$$E : \mathsf{FI} \to \mathcal{V}$$

and

$$F : \mathcal{V} \to \mathcal{W}$$

how do the Taylor coefficients of E and F (in the FI and Goodwillie sense respectively) relate to the coefficients of $F \circ E$? Is there some sort of chain rule analogous to that described by Greg Arone and Michael Ching in [AC11] that describes this relationship? Going in the other direction, given a functor

$$G : \mathsf{FI} \to \mathcal{W}$$

we can ask what the Taylor coefficients of E and G tell us about the Goodwillie tower of $\mathrm{Lan}_E\, G$. These same questions relating can be applied to the zoo of generalizations of FI-calculus.

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