Set Theory Syllabus

Kaya Arro

June - July 2023

Introduction

The primary goal of this course is to introduce students to the essence of academic mathematics: the application of the rules of logic to a set of axioms to deduce theorems. Mathematics has an intimidating reputation, and often even students who enjoy and excel in math classes in high school still have little exposure to the joys and rigors of academic mathematics. This course aims to teach students what academic mathematics is, that it is delightful, and that they can do it.

A first course on academic mathematics could feasibly focus on a range of topics, such as number theory, geometry, graph theory, or others. This course focuses on set theory for several reasons:

- Set theory is foundational.
  - Essentially all other mathematical disciplines can be embedded within set theory, in a sense that we will make in the course.
  - It can be exciting to see how any other mathematical fact can be deduced from and within set theory, even ones which we usually take for granted but which actually have genuine, formal, abstract proofs, such as the facts that addition and multiplication of natural numbers are associative and commutative.
  - Although set theory is only one of several viable foundations for mathematics, it is the one which is most widely used and which is most familiar to most working mathematicians.
- Set theory has a very abstract flavor. This can be intimidating, but it can also be appealing because of how it stimulates the imagination. Almost all of academic mathematics is rather abstract, and so set theory can be thought of as representing the qualitative, abstract nature of the field.
- Set theory is a beautiful and surprising subject in its own right. Starting from the simplest of premises – a list of rules allowing us to reason about collections of things – many startling and fantastical results emerge, such as the fact that there are a multitude of different types of infinity, each with their own properties.

Course Structure

Class time will be split between lecture time, including questions and answers and group discussions of the material, and group work, during which students will think about questions and problems together under the close and helpful guidance of the instructor. Mathematics is a collaborative, social project, and we will aim to emphasize its social nature in this course. Working collaboratively on problems has numerous pedagogical merits:

- Mathematical proofs usually require several steps. Sometimes one student will readily identify one step, but have trouble seeing another, while another student may complementary insights.
- Having a collaborator can make the problem-solving process go much more smoothly and significantly reduce the frustration of being stuck – both by making getting stuck less common, and by having someone to commiserate with when one does get stuck. During the group work portion of the course, the instructor will be present to help out any time that students are at a complete loss for how to proceed.
- Thinking about how to best explain an idea to a peer can help to sharpen one's own understanding of the material. Successfully explaining an idea to a peer can greatly bolster a student's confidence with the material.

Before we dive into the abstraction of set theory, we will begin with Peano arithmetic, an axiomatic foundation for the study of the natural numbers that will allow us to give rigorous proofs of properties of arithmetic that we take for granted from elementary school. In years past, students have greatly enjoyed seeing a familiar and even mundane topic in a bizarre new light. Students who are interested may also try out the Natural Numbers Game, an introduction to writing computer-readable proofs in Lean using Peano Arithmetic.

As we begin set theory proper, much of the course will follow William Weiss' set theory textbook *An Introduction to Set Theory*, specifically chapters 0-7 and possibly chapters 8-10. Students are encouraged but not required to consult the text outside of class. There is no need to buy anything: the book can be legally downloaded as a pdf from the foregoing link. However, if a physical copy is desired, one can be purchased very cheaply.

Students will be given homework questions based on questions that arise in each class. Students are strongly encouraged to collaborate on the homework, but this should not mean splitting up the problems. Each student should understand as completely as possible the solution to each problem. Students are also encouraged to discuss the problems in office hours with the course TA. Students will be asked to share their progress in class, but if students find certain homework problems to be too challenging, they need not complete every single one, but they should give each assignment their best effort. The homework is meant to give students necessary practice with the material. Its primary function is *not* to be an assessment. Instead, students will be assessed based on their overall engagement with the course material.

Outline of Content

- Logic
  - Propositional logic
  - Predicate logic
  - Formal systems
- Peano arithmetic
  - Basic facts, induction
  - Addition
  - Multiplication
  - Exponentiation
  - Associativity proofs
  - Commutativity proofs
- The axioms of ZF set theory and preliminary theorems
- Important definitions
  - Operations on sets
  - Functions and their properties
  - Cardinality
  - Relations and orders
- Ordinal numbers
- The Axiom of Choice and more about cardinality
- Further topics, depending on time and student interest
  - Cardinal arithmetic
  - Additional, non-standard axioms, including universes
  - The real numbers: a first look at analysis
  - A glimpse of further topics in elementary number theory
  - Possibly other topics depending on student interest