Corepresentation Functor Calculus

JMM 2025 Homotopy Theory Special Session

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These slides are available at kayaarro.site/pdf/JMM25.pdf
in case you'd like to follow along on your own machine.

The talk

Today I'm going to tell you about a recipe for cooking up new functor calculi. In more detail, here's how my talk will proceed:

1. The previous (title) slide
2. This slide (table of contents?)
3. What are functor calculi?
4. Backstory: the original corepresentation calcul(us/i)
5. Why invent new functor calculi?
6. Some results
7. A definition

I'm more invested in my talk being intelligible than in getting through all my slides, so if anything is unclear, don't wait until the end to ask questions!

What are functor calculi?

The term "functor calculus" encompasses a range of techniques that are superficially similar to one another but which do not all sit neatly under a single formal rubric. These include (most famously) the Goodwillie calculus of Goodwillie, the orthogonal calculus of Weiss, the embedding calculus of Goodwillie and Weiss, the abelian calculus of Johnson and McCarthy, and other calculi, all with various interrelationships.

What are functor calculi?

What these calculi have in common is that

1. for each $n \in \mathbb{N}$, they specify a family $\mathfrak{D}_n$ of diagrams in $\mathcal{C}$ and define a functor $F : \mathcal{C} \to \mathcal{D}$ to be n-polynomial if F sends diagrams in $\mathfrak{D}_n$ to limit diagrams in $\mathcal{D}$;

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   2.1 so that any functor $F : \mathcal{C} \to \mathcal{D}$ admits a universal n-polynomial approximation $F \to \mathbf{P}_n F$ and hence a Taylor tower $F \to \cdots \mathbf{P}_{n+1} F \to \mathbf{P}_n F \to \cdots \mathbf{P}_0 F$;

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   3.1 i.e. there is an equivalence $\mathrm{Hmg}_n(\mathcal{C}, \mathcal{D}) \simeq \{\text{Something nice}\}$.

The original corepresentation calcul(us/i)

(The predecessor to) this project originated in my interest in Weiss' orthogonal calculus, which is a bit of a strange beast. Orthogonal calculus studies functors $F : \mathcal{J} \to \mathcal{S}$, where $\mathcal{J}$ is the ∞-category of Euclidean spaces and $\mathcal{S}$ the ∞-category of topological spaces/simplicial sets/homotopy types/anima/whatever.

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I decided to try to better understand what makes orthogonal calculus tick by studying something that isn't orthogonal calculus. More specifically, I noticed that the category **FI** of finite sets and injections has certain things in common with $\mathcal{J}$. I like to think of **FI** as "Euclidean spaces for $\mathbb{F}_1$." And indeed **FI** supports a functor calculus!

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Q: What's up with that ambiguous "calcul(us/i)" spelling in the title of this slide?
A: Orthogonal calculus is very probably a corepresentation calculus, but it isn't proven to be yet! I believe that Greg Arone and Niall Taggart are working on a result that would imply that it is.

Why invent new functor calculi?

Well, I answered this slide's title on the last slide: **FI** is like a weird (read: cool and exciting) discrete cousin of $\mathcal{J}$. But why should anyone else care? And why keep going after **FI**-calculus? We have a corepresentation calculus at home!

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It turns out that **FI** is a protagonist of a story called "representation stability." And **FI**-calculus turns out to be "representation stability (of **FI**-modules) in the land of ∞-categories" (plus some new and shiny calculus-shaped baubles).

The hope is that by generalizing the techniques of **FI**-calculus to other domain categories, it may be possible to find ∞-categorical functor calculus extensions of other interesting instances of representation stability.

Taylor coefficients

Corepresentation calculi deal with functors $\mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is any stable presentable ∞-category.

For each $n \in \mathbb{N}$, there is an equivalence of ∞-categories $\mathrm{Hmg}_n(\mathcal{C}, \mathcal{D}) \simeq \mathit{Fun}(\mathcal{C}_n \setminus \mathcal{C}_{n-1}, \mathcal{D})$ for a certain family of sub-∞-categories

$$\mathcal{C}_0 \subseteq \cdots \subseteq \mathcal{C}_{n-1} \subseteq \mathcal{C}_n \subseteq \cdots \subseteq \mathcal{C}$$

These functors from $\mathcal{C}_n \setminus \mathcal{C}_{n-1}$ are the Taylor coefficients.

Aggregate cofficients

For $x \in \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$, the composite

$$* \xrightarrow{x} \mathcal{C}_n \setminus \mathcal{C}_{n-1} \xrightarrow{\mathbf{C}_n F} \mathcal{D}$$

determines a functor $\mathbf{C}_x : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathcal{D}$. We define $\mathrm{Coeff}_{\mathcal{C}}$ to be the full subcategory of $\mathrm{Fun}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}), \mathcal{D})$ spanned by the $\mathbf{C}_x$s and obtain the aggregate Taylor coefficient functor $\mathbf{C} : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathrm{Coeff}_{\mathcal{C}}, \mathcal{D})$.

Propagation

Theorem (A.)

If $\mathcal{C}$ is equipped with a corepresentation calculus and $\varpi : \mathcal{E} \to \mathcal{C}$ is a Cartesian fibration, there is a lift of the calculus on $\mathcal{C}$ to a calculus on $\mathcal{E}$.

Theorem (A.)

If $\mathcal{C} \simeq$ FI and ϖ is a right fibration, we find that $\mathcal{C} \simeq \mathrm{Coeff}_{\mathcal{C}}$.

Examples

$\mathcal{M}fld_d$, the ∞-category of smooth d-dimensional manifolds and embeddings, admits a corepresentation calculus. Using the first result from the previous slide, we can propagate these examples to the following additional examples:

- **OI**, the category of totally ordered finite sets and order-preserving injections
- **CI**, the category of cyclically ordered finite sets and order-preserving injections
- $\mathbf{Braid}_M \overset{\text{def}}{=} \mathrm{FI} \downarrow_{\mathcal{M}fld} M$, the ∞-category of configurations of marked points in M and braids in M
- $\mathcal{M}fld_d^{\mathrm{Fr}}$, the ∞-category of d-dimensional framed manifolds and frame-preserving embeddings
- $\mathcal{M}fld_{2d}^{\mathrm{Sym}}$, the ∞-category of $2d$-dimensional symplectic manifolds and embeddings preserving the symplectic forms

etc...

The axioms

We define a corepresentation functor calculus to be an ∞-category $\mathcal{C}$ and a family $\{\mathfrak{D}_i\}_{i\in\mathbb{N}}$ of families of diagrams in $\mathcal{C}$ satisfying the following three axioms.

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- For $\mathcal{I}$ the domain of a diagram in some $\mathfrak{D}_i$, $\mathcal{I}$ has initial and terminal objects. An $\mathcal{I}$-diagram in a stable ∞-category is a limit diagram if and only if it is a colimit diagram.

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- A filtration axiom on the next slide.

The filtrations

Denote by $\mathcal{C}_i$ the full sub-∞-category of $\mathcal{C}$ of objects $c \in \mathcal{C}$ such that $\Sigma^\infty_+ \mathcal{C}(c, -)$ sends diagrams in $\mathfrak{D}_i$ to limit diagrams.

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Denote by $\mathcal{C}_i^0 \stackrel{\text{def}}{=} \mathcal{C}_i$ and $\mathcal{C}_i^{n+1}$ the full sub-∞-category of objects that are either in $\mathcal{C}_i^n$ or are the terminal object of some diagram in $\mathfrak{D}_i$ that otherwise takes values in $\mathcal{C}_i^n$.

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- We require that $\mathcal{C} = \bigcup_n \mathcal{C}_i^n$ for each $i \in \mathbb{N}$.

Intuitively, we think of $\mathcal{C}$ as being “generated” by $\mathcal{C}_i$ and diagrams in $\mathfrak{D}_i$.

Applying the axioms

We call a functor $E : \mathcal{C} \to \mathcal{D}$ n-polynomial if it sends all diagrams in $\mathfrak{D}_n$ to limit diagrams. The first axiom ensures that we actually obtain a Taylor tower. The second and third axioms give us the following and justify calling corepresentation functor calculi functor calculi.

Theorem (A.)

There are equivalences of ∞-categories

$$\mathrm{Poly}_n\mathcal{V} \cong \mathrm{Fun}(\mathcal{C}_n, \mathcal{D})$$
$$\mathrm{Hmg}_n\mathcal{V} \cong \mathrm{Fun}(\mathcal{C}_n \setminus \mathcal{C}_{n-1}, \mathcal{D})$$

The first of these equivalences is a distinctive feature of corepresentation functor calculi.

Cubes

The first two axioms (i.e. everything except the filtration axiom) are automatically satisfied in the very typical case that the diagrams of $\mathfrak{D}_n$ are $n+1$-cubes and if an n-cube X is a face of an $n+1$-cube in $\mathfrak{D}_n$, then $X \in \mathfrak{D}_{n-1}$. This leaves only the filtration axiom to check "by hand."

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Weiss' orthogonal calculus notably does not use cubes. Conjecturally, orthogonal calculus is a corepresentation functor calculus, but the second axiom has not yet been verified.

A conjecture (work in progress)

Conjecture

When $\varpi : \mathcal{D} \to \mathbf{FI}$ is as above, $\mathcal{D}$ is a 1-category with finite automorphism groups, and Tate cohomology vanishes in $\mathcal{D}$ (e.g. when $\mathcal{D}$ is $\mathbb{Q}$-linear), $\mathbf{C}E$ recovers the Taylor tower of E, as is the case when $\mathcal{D} = \mathbf{FI}$.