Axioms for functor calculus
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Lifting Functor calculus

AMS 2024 Spring Southeastern Sectional Meeting

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These slides are available at kayaarro.site/pdf/FSU.pdf in case you'd like to follow along on your own machine.

Overview

We work with ∞-categories, but if you are comfortable with 1-categories and not ∞-categories, everything should still be intelligible (even if a few things work a little differently).

- We review functor calculus.
- We discuss FI-calculus, the motivation for the present work.
- By scrutinizing the arguments of FI-calculus, we extract an axiomatization for a family of functor calculi with similar qualities.
- We observe that functor calculus can be lifted along cartesian fibrations, furnishing us with a rich family of examples of functor calculi.

Ask questions freely!

The disunity of functor calculi

The term “functor calculus” refers to a family of similar techniques for studying functors. Examples include Goodwillie calculus, Goodwillie–Weiss embedding calculus, Weiss orthogonal calculus, Johnson–McCarthy abelian functor calculus, etc.

Some functor calculi fit together under a common umbrella, e.g. current work of Hess and Johnson, but there is not much of a unifying framework for all functor calculi beyond the fact that they share certain features, which we recall on the next slide.

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One of the objectives of this talk is to describe an axiomatization of a new family of functor calculi modelled after FI-calculus and, conjecturally, orthogonal calculus.

Features of functor calculi

Generally, a functor calculus involves fixing ∞-categories $\mathcal{C}$ and $\mathcal{D}$ and considering functors $\mathcal{C} \to \mathcal{D}$.

- For $n \in \mathbb{N}$, there is a particular family $\mathfrak{D}_n$ of diagrams (whose indexing ∞-categories have initial objects) in $\mathcal{C}$. A functor $E : \mathcal{C} \to \mathcal{D}$ is called n-polynomial or n-excisive if E sends all diagrams in $\mathfrak{D}_n$ to limit diagrams in $\mathcal{D}$.
- Every functor $E : \mathcal{C} \to \mathcal{D}$ admits a universal approximation $E \to \mathbf{P}_n E$ by an n-polynomial functor.
- For $m \geq n$, every n-polynomial functor is m-polynomial; as a consequence, one obtains a Taylor tower

$$E \to \cdots \to \mathbf{P}_n E \to \cdots \to \mathbf{P}_0 E$$

Taylor coefficients

The first payoff of this structure is a family of invariants for functors $\mathcal{C} \to \mathcal{D}$.

A functor $E : \mathcal{C} \to \mathcal{D}$ is called *n*-homogeneous E is n-polynomial and $\mathbf{P}_{n-1}E \cong 0$. The ∞-category of n-homogeneous functors is typically equivalent to the ∞-category of objects of $\mathcal{D}$ equipped with some extra structure (e.g. the action of some (∞-)group).

The layers $\mathbf{D}_n E \overset{\text{def}}{=} \operatorname{fib}(\mathbf{P}_n E \to \mathbf{P}_{n-1} E)$ are n-homogeneous. The Taylor coefficients of E are the $\mathcal{D}$-objects-with-structure corresponding to these layers. Often, the Taylor coefficients of E considered in aggregate carry some further structure from which one may hope to recover information about E.

FI-calculus

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The functor calculus on which our axioms are modelled is FI-calculus, which studies functors $\mathrm{FI} \to \mathcal{V}$ where FI is the category of finite sets and injections and $\mathcal{V}$ is a presentable[1] stable[2] ∞-category. FI-calculus is an ∞-categorification of representation stability, but that's beside the point at the moment.

[1]This is a tameness condition ensuring $\mathcal{V}$ is generated by "small" data and cocomplete.
[2]This is roughly the ∞-categorical analog of an abelian category.

Properties of FI-calculus

For us, the salient features of FI-calculus are these:

- The ∞-category $\mathcal{V}$ must be stable.
- The diagrams in $\mathfrak{D}_n$ are indexed by $\mathcal{P}(n+1)$, the powerposet of $n+1$.
- $\mathcal{P}(n+1)$ has a terminal object.
- A $\mathcal{P}(n+1)$-diagram in a stable ∞-category is a limit diagram if and only if it is a colimit diagram.

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- A $\mathcal{P}(n+1)$-diagram in a stable ∞-category is a limit diagram if and only if it is a colimit diagram.
- FI admits a filtration by subcategories $\mathsf{FI}_{\leq n}$ and n-homogeneous functors are classified by functors $(\mathsf{FI}_{\leq n} \setminus \mathsf{FI}_{\leq n-1}) \to \mathcal{V}$.[3]
- The Taylor coefficients of a functor admit natural structure maps endowing the Taylor coefficients themselves with the structure of a functor $\mathsf{FI} \to \mathcal{V}$.

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The axioms

We define a corepresentation functor calculus to be an ∞-category $\mathcal{C}$ and a family $\{\mathfrak{D}_i\}_{i \in \mathbb{N}}$ of families of diagrams in $\mathcal{C}$ satisfying the following three axioms.

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- For $\mathcal{I}$ the domain of a diagram in some $\mathfrak{D}_i$, $\mathcal{I}$ has initial and terminal objects. An $\mathcal{I}$-diagram in a stable ∞-category is a limit diagram if and only if it is a colimit diagram.

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- For $\mathcal{I}$ the domain of a diagram in some $\mathfrak{D}_i$, $\mathcal{I}$ has initial and terminal objects. An $\mathcal{I}$-diagram in a stable ∞-category is a limit diagram if and only if it is a colimit diagram.
- A filtration axiom on the next slide.

The filtrations

Denote by $\mathcal{C}_i$ the full sub-∞-category of $\mathcal{C}$ of objects $c \in \mathcal{C}$ such that $\Sigma^{\infty}_{+}\mathcal{C}(c,-)$ sends diagrams in $\mathfrak{D}_i$ to limit diagrams.

The filtrations

Denote by $\mathcal{C}_i$ the full sub-∞-category of $\mathcal{C}$ of objects $c \in \mathcal{C}$ such that $\Sigma^\infty_+ \mathcal{C}(c, -)$ sends diagrams in $\mathfrak{D}_i$ to limit diagrams.

Denote by $\mathcal{C}_i^0 \stackrel{\text{def}}{=} \mathcal{C}_i$ and $\mathcal{C}_i^{n+1}$ the full sub-∞-category of objects that are either in $\mathcal{C}_i^n$ or are the terminal object of some diagram in $\mathfrak{D}_i$ that otherwise takes values in $\mathcal{C}_i^n$.

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- We require that $\mathcal{C} = \bigcup_n \mathcal{C}_i^n$ for each $i \in \mathbb{N}$.

Intuitively, we think of $\mathcal{C}$ as being "generated" by $\mathcal{C}_i$ and diagrams in $\mathfrak{D}_i$.

Applying the axioms

We call a functor $E : \mathcal{C} \to \mathcal{V}$ n-polynomial if it sends all diagrams in $\mathfrak{D}_n$ to limit diagrams. The first axiom ensures that we actually obtain a Taylor tower. The second and third axioms give us the following and justify calling corepresentation functor calculi functor calculi.

Theorem (A.)

There are equivalences of ∞-categories

$$\mathrm{Poly}_n\mathcal{V} \cong \mathrm{Fun}(\mathcal{C}_n, \mathcal{V})$$
$$\mathrm{Hmg}_n\mathcal{V} \cong \mathrm{Fun}(\mathcal{C}_n \setminus \mathcal{C}_{n-1}, \mathcal{V})$$

The first of these equivalences is a distinctive feature of corepresentation functor calculi.

Cubes

The first two axioms (i.e. everything except the filtration axiom) are automatically satisfied in the very typical case that the diagrams of $\mathfrak{D}_n$ are indexed by $\mathcal{P}(n+1)$ – i.e. the diagrams are "$n+1$-cubes" – and, whenever an n-cube X in $\mathcal{C}$ is a face of an $n+1$-cube in $\mathfrak{D}_n$, $X \in \mathfrak{D}_{n-1}$. This leaves only the filtration axiom to check "by hand," but as we will soon see, we may also get this axiom "for free."

Cubes

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Weiss' orthogonal calculus notably does not use cubes. Conjecturally, orthogonal calculus is a corepresentation functor calculus, but the second axiom has not yet been verified.

Manifolds

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The ∞-category $\mathcal{M}fld_d$ of d-dimensional manifolds and embeddings is a corepresentation functor calculus when we set $\mathfrak{D}_i$ to be those $i+1$-cubes in $\mathcal{M}fld_d$ such that each point in the final manifold is hit at least i times by the $i+1$ subsets of cardinality i and the smaller sets are sent to the evident intersections. When $d = 0$, this recovers FI-calculus.

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The resulting functor calculus is in some ways dual to Goodwillie–Weiss embedding calculus. The functors one considers are covariant and the construction of the polynomial approximations is very different, with the approximations being taken from the “other end of the ∞-category.”

Cartesian fibrations

Recall that a cartesian fibration is a functor $\varpi : \mathcal{D} \to \mathcal{C}$ such that given a morphism $f : c' \to c$ in $\mathcal{C}$ and a lift d of c, there is a universal lift $g : d' \to d$ of f.

I like to think of this definition as saying: given $c \in \mathcal{C}$, a $\mathcal{D}$-structure on c, and a subobject c' of c, there is a canonical restriction of the $\mathcal{D}$-structure on c to a $\mathcal{D}$-structure on the subobject c'.

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Theorem (A.)

Suppose $\mathcal{C}$ satisfies the required axioms, and let

$$\varpi : \mathcal{D} \to \mathcal{C}$$

be a Cartesian fibration. Then $\mathcal{D}$ inherits a corepresentation functor calculus from $\mathcal{C}$.

Bonus results

Recall that a right fibration is a conservative cartesian fibration, i.e. one whose fibers are ∞-groupoids.

Theorem (A.)

Let $\varpi : \mathcal{D} \to \mathsf{FI}$ be a right fibration. Then, as with FI, the Taylor coefficients of a functor $E : \mathcal{D} \to \mathcal{V}$ naturally assemble into a functor $\mathbf{C}E : \mathcal{D} \to \mathcal{V}$.

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Conjecture

When $\varpi : \mathcal{D} \to \mathsf{FI}$ is as above, $\mathcal{D}$ is a 1-category with finite automorphism groups, and Tate cohomology vanishes in $\mathcal{V}$ (e.g. when $\mathcal{V}$ is $\mathbb{Q}$-linear), $\mathbf{C}E$ recovers the Taylor tower of E, as is the case when $\mathcal{D} = \mathsf{FI}$.

Example: braids

Fix a manifold M and let $\mathbf{Braid}_M \stackrel{\text{def}}{=} \mathsf{FI} \downarrow M$ – i.e. the category in which objects are configurations of marked points in M and morphisms are braids in M.

The forgetful functor $\mathbf{Braid}_M \to \mathsf{FI}$ is a right fibration, so $\mathbf{Braid}_M$ admits a corepresentation functor calculus in which the Taylor coefficients carry the structure of a functor $\mathbf{Braid}_M \to \mathcal{V}$.

More examples

When $M = \mathbb{R}^2$, the automorphism groups in $\mathbf{Braid}_M$ are the braid groups. When $M = S^1$, $\mathbf{Braid}_M$ is the category of finite cyclically ordered sets and cyclic-order-preserving injections. When $M = \mathbb{R}^1$, $\mathbf{Braid}_M$ is the category of totally ordered finite sets and order-preserving injections.

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More generally, cartesian fibrations over $\mathcal{M}fld_d$ tend to look like manifolds equipped with "local structure." For example, we may take an ∞-category of manifolds equipped with some tangent structure, such as an orientation, a framing, a metric, etc.

The End