FI-calculus and representation stability

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Conventions and notation

- All category stuff is ∞-category stuff
- FI is the category of finite sets and injections. $\mathsf{FI}_{\leq n}$ is the full subcategory spanned by sets of cardinality at most n.
- $\mathcal{V}$ is an arbitrary, fixed stable presentable category.
  - Feel free to take $\mathcal{V} = \mathcal{S}p$ or $\mathrm{Ch}\mathcal{A}b$.
- FI$\mathcal{V}$ is the category of functors FI $\to \mathcal{V}$. Its objects are called FI-objects.
- $\mathfrak{S}_n$ is the group of permutations of an n-element set.

Overview

In analogy to Weiss' orthogonal calculus, we introduce a functor calculus for functors

$$\mathsf{FI} \to \mathcal{V}$$

We define a Taylor tower and show that n-homogeneous FI-objects are classified by $\mathfrak{S}_n$-objects, allowing us to define the Taylor coefficients of an FI-object.

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We show that FI-calculus is a generalization of representation stability for FI-modules.

Excision

Definition

A functor $C : \mathcal{P}(n) \to \mathrm{FI}$ is a standard n-cube if there exists $S \in \mathrm{FI}$ such that $C(T) = S \sqcup T$

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A functor $E : \mathsf{FI} \to \mathcal{V}$ is n-polynomial if it sends every standard $n+1$-cube to a Cartesian cube (i.e. a limit diagram). We denote the category of such $\mathrm{Poly}_n\mathcal{V}$.

Left Kan extension (presented in finite degree)

Theorem

An FI*-object E is n-polynomial if and only if*

$$E \cong \mathsf{Lan}_{\mathsf{FI}_{\leq n}}^{\mathsf{FI}} \mathsf{Res}_{\mathsf{FI}}^{\mathsf{FI}_{\leq n}} E$$

That is, we have and equivalence of categories

$$\mathsf{Res}_{\mathsf{FI}}^{\mathsf{FI}_{\leq n}} : \mathrm{Poly}_n \mathcal{V} \simeq \mathsf{FI}_{\leq n}\mathcal{V} : \mathsf{Lan}_{\mathsf{FI}_{\leq n}}^{\mathsf{FI}}$$

Formal Taylor towers

Corollary

$\mathrm{Poly}_n\mathcal{V}$ is both reflective and coreflective in $\mathsf{FI}\mathcal{V}$. We denote the reflection functor $\mathbf{P}_n$ and the coreflection functor $\mathbf{Q}_n$.

Formal Taylor towers

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Definition

A functor $E : \mathbb{N}^{\mathrm{op}} \to \mathsf{FI}\mathcal{V}$ is a formal Taylor tower if $\mathbf{P}_i E_{i+1} \xrightarrow{\sim} E_i$. We denote the category of such $\mathrm{FTT}\mathcal{V}$.

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Definition

We have a functor

$$\mathbf{P} \overset{\mathrm{def}}{=} (E \mapsto \{\mathbf{P}_i E\}) : \mathsf{FI}\mathcal{V} \to \mathrm{FTT}\mathcal{V}$$

We call $\mathbf{P}E$ the Taylor tower of E.

Analytic and convergent

Definition

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We call a formal Taylor tower convergent if it is a colimit of eventually constant formal Taylor towers, or, equivalently, if it is a Taylor tower. We denote the category of such $\mathrm{FTT}\mathcal{V}^{\mathrm{Conv}}$.

An equivalence

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$$\mathsf{P} : \mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \simeq \mathrm{FTT}\mathcal{V}^{\mathrm{Conv}} : \mathsf{lim}$$

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$$\mathbf{P} : \mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \simeq \mathrm{FTT}\mathcal{V}^{\mathrm{Conv}} : \mathsf{lim}$$

This fact is a formal consequence of the facts that $\mathsf{FI}\mathcal{V}$ is stable, that $\mathrm{Poly}_n\mathcal{V}$ are each both reflective and coreflective, and that $\mathsf{FI}\mathcal{V}$ is generated by polynomial objects.

The next slide is a glimpse at the core of the proof. $\mathbf{J}_n$ is coreflection into n-polynomial formal Taylor towers.

The proof

Proof.

$$\begin{aligned}
\mathbf{P}\left(\lim_{i\in\mathbb{N}^{\mathrm{op}}} E_i\right) &\cong \mathbf{P}\left(\operatorname{colim}_{n\in\mathbb{N}} \mathbf{Q}_n \lim_{i\in\mathbb{N}^{\mathrm{op}}} E_i\right) \\
&\cong \mathbf{P}\left(\operatorname{colim}_{n\in\mathbb{N}} \lim_{i\in\mathbb{N}^{\mathrm{op}}} (\mathbf{J}_n E)_i\right) \\
&\cong \operatorname{colim}_{n\in\mathbb{N}} \mathbf{P}\left(\lim_{i\in N} (\mathbf{J}_n E)_i\right) \\
&\cong \operatorname{colim}_{n\in\mathbb{N}} (\mathbf{J}_n E)_i \\
&\cong E
\end{aligned}$$

$\square$

Homogeneous FI-objects

Definition

An FI-object E is *n-homogeneous* if $E \in \mathrm{Poly}_n\mathcal{V}$ and $\mathbf{P}_{n-1}E \cong 0$. We denote the category of such $\mathrm{Hmg}_n\mathcal{V}$.
We define

$$\mathbf{D}_n \stackrel{\text{def}}{=} \mathsf{fib}\,(\mathbf{P}_n \to \mathbf{P}_{n-1})$$

and call $\mathbf{D}_nE$ the *nth homogeneous layer* of E.

Cohomogeneous FI-objects

Definition

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Theorem

$$\mathrm{Hmg}_n\mathcal{V} \simeq \mathrm{coHmg}_n\mathcal{V} \simeq \mathfrak{S}_n\mathcal{V}$$

Classification of homogeneous FI-objects

Proof.

Given $E \in \mathrm{Hmg}_n\mathcal{V}$, each row and each column of the following diagram is a fiber sequence.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathbf{D}_n\mathbf{R}_n E \arrow[d] \\
\mathbf{Q}_{n-1}E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathbf{R}_n E \arrow[d] \\
\mathbf{Q}_{n-1}E \arrow[r] & 0 \arrow[r] & \mathbf{P}_{n-1}\mathbf{R}_n E
\end{tikzcd}$$

This establishes that $E \cong \mathbf{D}_n\mathbf{R}_n$, and a dual argument gives the opposite direction. □

Coefficients, representables

Definition

For $E \in \mathrm{FI}\mathcal{V}$ we denote by $\mathbf{C}E(n)$ the $\mathfrak{S}_n$-object corresponding to $\mathbf{D}_n E$ and we call $\mathbf{C}E(n)$ the nth Taylor coefficient of E. We define Taylor coefficients of formal Taylor towers in the same fashion.

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Definition

For $X \in \mathcal{V}$, define

$$F_{n,X} \overset{\text{def}}{=} k \mapsto \mathrm{FI}(n,k) \otimes X : \mathrm{FI} \to \mathcal{V}$$

Feel free to think of $\mathcal{V} = \mathcal{S}p$ and $X = \mathbb{S}$ so that $F_{n,X} = \Sigma^\infty_+ \mathrm{FI}(n,-)$.

Coefficients of representables

Theorem

$$\mathbf{C}F_{X,n}(k) \cong \mathrm{FI}(k, n) \pitchfork X$$

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For $E \in \mathsf{FI}\mathcal{V}$ or $E \in \mathrm{FTT}\mathcal{V}$, $\mathbf{C}E$ is an FI*-object!*

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Theorem

Suppose that the Tate construction vanishes for $\mathfrak{S}_n$-objects in $\mathcal{V}$, as when $\mathcal{V} = \mathcal{S}p^{\mathbb{Q}}$. Then

$$\mathbf{C} : \mathrm{FTT}\mathcal{V} \simeq \mathsf{FI}\mathcal{V}$$

Representation stability: a refresher

Recall that irreducible representations of $\mathfrak{S}_n$ over a field of characteristic 0 are in bijection with partitions of n (non-increasing sequences of positive integers summing to n).

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Given an $\mathfrak{S}_n$-representation V corresponding to a partition $(k_1, \ldots, k_j)$ and $m \in \mathbb{N}$, we associate the $\mathfrak{S}_{n+m}$-representation corresponding to the partition $(k_1 + m, \ldots, k_j)$.

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Representation stability is a property enjoyed by many naturally-occurring FI-modules which ensures that the irreducible decompositions of the representations of the automorphism groups of FI "stabilize" in the suggested sense on sufficiently large sets.

Representation stability: the key calculation

Theorem

For $\mathcal{V} = \mathcal{S}p$,

$$\mathbf{D}_n F_{\mathbb{S},n}(k) \cong \Sigma^{\infty-n} S_* BP(n,k)$$

where S_* is unreduced suspension that makes one cone point a base point, B is the classifying space of a category, and $P(n,k)$ is the poset of non-empty partial bijections between (n) and (k).

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Theorem

For $k \geq 2n-1$, $\mathbf{D}_n F_{\mathbb{S},n}(k)$ is a wedge of copies of $\mathbb{S}$.

Representation stability

Theorem

Suppose $E : \mathsf{FI} \to \mathbb{Q}\mathbf{Vect}$ *is representation stable. Then*

$$\mathrm{fib}\left(HE \to \lim_{i \in \mathbb{N}^{\mathrm{op}}} (\mathbf{P}HE)_i\right)$$

has finite support – i.e. HE *is polynomial to within finite error.* H *means "Eilenberg-MacLane spectrum."*

Suppose $E : \mathsf{FI} \to \mathcal{S}p^{\mathbb{Q}}$ *is polynomial. Then* $H_i E$ *is representation stable for all* $i \in \mathbb{Z}$*.* H_i *means "homology."*

Generalization to manifolds

$\mathsf{FI} = \mathbf{Embd}_0$ where $\mathbf{Embd}_d$ is the category of d-dimensional smooth manifolds and embeddings. The same techniques allow for a functor calculus for functors

$$\mathbf{Embd}_d \to \mathcal{V}$$

in which the n-homogeneous objects are classified by functors

$$\mathrm{Disk}_d^{\sqcup n} \to \mathcal{V}$$

where $\mathrm{Disk}_d^{\sqcup n}$ is the full subcategory of $\mathbf{Embd}_d$ spanned by the disjoint union of n d-disks. This is related to, but distinct from, factorization homology and the manifold calculus of Goodwillie-Weiss.

Generalization to Cartesian fibrations

Let

$$\varpi : \mathcal{D} \to \mathbf{Embd}_d$$

be a Cartesian fibration. The same techniques allow for a calculus for functors $\mathcal{D} \to \mathcal{V}$ in which the n-homogeneous objects are classified by functors

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Example

Fix a manifold M and let $\mathbf{Braid}_M \stackrel{\text{def}}{=} \mathsf{FI} \downarrow M$ – i.e. the category in which objects are configurations of marked points in M and morphisms are braids. Then $\mathbf{Braid}_M$ admits a functor calculus analogous to FI-calculus in which homogeneous objects are classified by actions of the braid groups of M on objects of $\mathcal{V}$.

Monoidicity

Suppose that $\mathcal{D}$ possesses an E_2-structure compatible with the Cartesian fibration ϖ. Then we obtain a functor calculus in which the degrees of excision are indexed by finite multisets of objects of $\mathcal{D}$.

The End