Representation stability and functor calculus

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Conventions and notation

- $\mathcal{V}$ is an arbitrary, fixed stable presentable ∞-category.
- FI is the category with objects finite sets and morphisms injections. $\mathrm{FI}_{\leq n}$ is the full subcategory of FI spanned by set of cardinality at most n. $\mathfrak{S}_n$ is the nth symmetric group.
- $\mathrm{FI}\mathcal{V}$ is the ∞-category of functors $\mathrm{FI} \to \mathcal{V}$. Its objects are called FI-objects. $\mathrm{FI}_{\leq n}\mathcal{V}$ and $\mathfrak{S}_n\mathcal{V}$ are defined similarly.
- For $X \in \mathcal{V}$ and S a set, $S \otimes X$ denotes the S-fold coproduct X and $S \pitchfork X$ the S-fold product of X.

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- We describe a functor calculus for FI-objects. We define a Taylor tower and show that n-homogeneous FI-objects are classified by $\mathfrak{S}_n$-objects, allowing us to define the Taylor coefficients of an FI-object.
- We show that FI-calculus categorifies representation stability.
- We describe natural transformations between Taylor coefficients and show that Taylor towers are recovered from these natural transformations up to the vanishing of a Tate construction.

Representations of symmetric groups

Recall that in characteristic 0, isomorphism classes of $\mathfrak{S}_n$-irreducibles are in bijection with partitions of n, where a partition of n is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\sum_i \lambda_i = n$.

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Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, and $m \geq n$, we define a partition λ^m of m by $\lambda^m \stackrel{\text{def}}{=} (\lambda_1 + m - n, \dots, \lambda_k)$.

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By Maschke's Theorem, a finite-dimensional $\mathfrak{S}_n$-representation is determined by an $\mathbb{N}$-linear combination of partitions of n, so we can extend this operation: given an $\mathfrak{S}_n$-representation V, we obtain an $\mathfrak{S}_m$-representation $V^{\uparrow m}$. In general, $V^{\uparrow m}$ is a subrepresentation of $\mathrm{Ind}_{\mathfrak{S}_n}^{\mathfrak{S}_m} V$.

FI-modules

For R a ring, we call a functor $E\colon \mathrm{FI} \to \mathcal{M}od_R$ an FI-module.

The endomorphism monoids of FI are the symmetric groups, so any FI-module restricts an $\mathfrak{S}_n$-representation for each n. A happy fact is that for many FI-modules E of natural interest, the representations $E(n)$ are related in a way made precise on the following slide.

Representation stability

Definition (Church, Farb [CF13, Definition 2.3])

A FI-module E over a field of characteristic 0 is representation stable if there exists $N \in \mathbb{N}$ such that for all $m \geq n \geq N$:

- The map $\mathrm{Ind}_n^m E(n) \to E(m)$ is surjective.
- The map $E(n) \to \mathrm{Res}_m^n E(m)$ is injective.
- $E(m) \cong E(n)^{\uparrow m}$.

Examples

- $H^k(C_n(M), \mathbb{Q})$ for M a compact manifold of dimension $d \geq 2$ and $C_n(M)$ the space of configurations of n marked points in M (Church [Chu11]).
- $H^k(\mathcal{M}_g^n, \mathbb{Q})$ where $\mathcal{M}_g^n$ is the moduli space of genus g Riemann surfaces with n marked points (Jiménez Rolland [Jim11]).
- All sorts of others; see e.g. the excellent survey of Jiménez Rolland and Wilson, [JW22].

Finite presentation

Theorem (Church et al. [Chu+14, Theorem C])

Let E be an FI*-module over a field of characteristic 0. The following are equivalent:*

- *$E(n)$ is finite dimensional for all $n \in \mathbb{N}$ and E is representation stable.*
- *There exists some N such that $E(n)$ is finite dimensional for $n \leq N$ and*

$$E \cong \mathrm{Lan}_{\mathsf{FI}_{\leq N}}^{\mathsf{FI}} \mathrm{Res}_{\mathsf{FI}}^{\mathsf{FI}_{\leq N}} E$$

Polynomial FI-objects

Definition

A functor $C : \mathcal{P}(n) \to \mathrm{FI}$ is a standard n-cube if there exists $S \in \mathrm{FI}$ such that $C(T) = S \sqcup T$

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Definition

A functor $E : \mathsf{FI} \to \mathcal{V}$ is n-polynomial if it sends every standard $n+1$-cube to a limit diagram. We denote the ∞-category of such $\mathrm{Poly}_n\mathcal{V}$.

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Corollary

$\mathrm{Poly}_n\mathcal{V}$ is both reflective and coreflective in $\mathsf{FI}\mathcal{V}$.

We denote the reflection functor $\mathbf{P}_n$ and the coreflection functor $\mathbf{Q}_n$.

Formal Taylor towers

Definition

We denote

$$\mathrm{FTT}\mathcal{V} \stackrel{\text{def}}{=} \lim \cdots \xrightarrow{\mathbf{P}_{n+1}} \mathrm{Poly}_{n+1}\mathcal{V} \xrightarrow{\mathbf{P}_n} \mathrm{Poly}_n\mathcal{V} \xrightarrow{\mathbf{P}_{n-1}} \cdots$$

We call the objects of $\mathrm{FTT}\mathcal{V}$ formal Taylor towers.

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Definition

We have a functor

$$\mathbf{P} \stackrel{\text{def}}{=} (E \mapsto \{\mathbf{P}_i E\}) \colon \mathsf{FI}\mathcal{V} \to \mathrm{FTT}\mathcal{V}$$

We call $\mathbf{P}E$ the Taylor tower of E.

Representation stability

Theorem

Suppose $E\colon \mathsf{FI} \to \mathbb{Q}\mathbf{Vect}$ *is representation stable. Then there is some* $n \in \mathbb{N}$ *such that*

$$\operatorname{fib}\left(HE \to \mathbf{P}_n HE\right)$$

has finite support – i.e. HE *is polynomial to within finite error.* H *means "Eilenberg-MacLane spectrum."*

Suppose $E\colon \mathsf{FI} \to \mathrm{Ch}\mathbb{Q}$ *is polynomial. Then* $H_i E$ *is representation stable for all* $i \in \mathbb{Z}$*.* H_i *means "homology."*

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We call an FI-object analytic if it is a limit of polynomial FI-objects. We denote the ∞-category of such $\mathrm{FI}\mathcal{V}^{\mathrm{Anly}}$.

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We call a formal Taylor tower convergent if it is a colimit of eventually constant formal Taylor towers, or, equivalently, if it is a Taylor tower. We denote the ∞-category of such $\mathrm{FTT}\mathcal{V}^{\mathrm{Conv}}$.

An equivalence

Theorem

$$\mathbf{P}\colon \mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \simeq \mathrm{FTT}\mathcal{V}^{\mathrm{Conv}} \colon \mathsf{lim}$$

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This fact is a formal consequence of the facts that $\mathsf{FI}\mathcal{V}$ is stable, that $\mathrm{Poly}_n\mathcal{V}$ are each both reflective and coreflective, and that $\mathsf{FI}\mathcal{V}$ is generated under colimits by polynomial objects.

Homogeneous FI-objects

Definition

An FI-object E is *n-homogeneous* if $E \in \mathrm{Poly}_n\mathcal{V}$ and $\mathbf{P}_{n-1}E \cong 0$. We denote the ∞-category of such $\mathrm{Hmg}_n\mathcal{V}$.
We define

$$\mathbf{D}_n \overset{\mathrm{def}}{=} \mathsf{fib}\,(\mathbf{P}_n \to \mathbf{P}_{n-1})$$

and call $\mathbf{D}_nE$ the *nth homogeneous layer* of E.

Cohomogeneous FI-objects

Definition

An FI-object E is *n-cohomogeneous* if $E \in \mathrm{Poly}_n\mathcal{V}$ and $\mathbf{Q}_{n-1}E \cong 0$. We denote the ∞-category of such $\mathrm{coHmg}_n\mathcal{V}$.

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Theorem

$$\mathrm{Hmg}_n\mathcal{V} \simeq \mathrm{coHmg}_n\mathcal{V} \simeq \mathfrak{S}_n\mathcal{V}$$

Classification of homogeneous FI-objects

Proof.

Given $E \in \mathrm{Hmg}_n\mathcal{V}$, each row and each column of the following diagram is a fiber sequence.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathbf{D}_n\mathbf{R}_nE \arrow[d] \\
\mathbf{Q}_{n-1}E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathbf{R}_nE \arrow[d] \\
\mathbf{Q}_{n-1}E \arrow[r] & 0 \arrow[r] & \mathbf{P}_{n-1}\mathbf{R}_nE
\end{tikzcd}$$

This establishes that $E \cong \mathbf{D}_n\mathbf{R}_n$, and a dual argument gives the opposite direction. □

Taylor coefficients

Definition

For $E \in \mathsf{FI}\mathcal{V}$ we denote by $\mathbf{C}E(n)$ the $\mathfrak{S}_n$-object corresponding to $\mathbf{D}_n E$ (this is $\mathbf{R}_n\mathbf{D}_n E(n)$) and we call $\mathbf{C}E(n)$ the nth Taylor coefficient of E. We define Taylor coefficients of formal Taylor towers in the same fashion.

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The Taylor coefficients $\mathbf{C}E(n)$ can be calculated directly from E in terms of certain cross-effects. In forthcoming work, Bridget Schreiner shows that these cross effects are themselves useful tools for calculating the homology of FI-spaces, especially those with representation stable homology.

Representables

Definition

For $X \in \mathcal{V}$, define

$$F_{n,X} \stackrel{\text{def}}{=} k \mapsto \mathrm{FI}(n,k) \otimes X \colon \mathrm{FI} \to \mathcal{V}$$

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Theorem

$$\mathbf{C}F_{X,n}(k) \cong \mathsf{FI}(k,n) \pitchfork X$$

Coefficients of representables

Corollary

For $E \in \mathsf{FI}\mathcal{V}$ *or* $E \in \mathrm{FTT}\mathcal{V}$, $\mathbf{C}E$ *is an* FI*-object!*

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Theorem

Suppose that the Tate construction vanishes for $\mathfrak{S}_n$-objects in $\mathcal{V}$, as when $\mathcal{V} = \mathcal{S}p^{\mathbb{Q}}$. Then

$$\mathbf{C}\colon \mathrm{FTT}\mathcal{V} \simeq \mathsf{FI}\mathcal{V}$$

Representation stability redux: the key calculuation

Theorem

For $\mathcal{V} = \mathcal{S}p$,

$$\mathbf{D}_n F_{\mathbb{S},n}(k) \cong \Sigma^{\infty-n} S_* BP(n,k)$$

where S_* is unreduced suspension that makes one cone point a base point, B is the classifying space of a poset, and $P(n,k)$ is the poset of non-empty partial bijections between (n) and (k).

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As an eventual result of this, the representations appearing in the stable range of a representation stable FI-module can be directly read off from its Taylor coefficients.

A general framework

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Let $\mathcal{C}$ be an ∞-category and $\{\mathfrak{D}_i\}$ a sequence of families of diagrams (e.g. cubes) in $\mathcal{C}$ satisfying a number of properties

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We obtain a Taylor tower as well as a classification of n-homogeneous functors in terms of Taylor coefficients

$$\mathcal{C}_n \setminus \mathcal{C}_{n-1} \to \mathcal{V}$$

The axioms

- For $\mathcal{I}$ the domain of a diagram in some $\mathfrak{D}_i$, $\mathcal{I}$ has initial and terminal objects and $\mathcal{I}$-diagrams in stable ∞-categories are limit diagrams if and only if they are colimit diagrams.

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- Denote by $\mathcal{C}_i^0 \stackrel{\text{def}}{=} \mathcal{C}_i$ and $\mathcal{C}_i^{n+1}$ the full sub-∞-category of objects that are either in $\mathcal{C}_i^n$ or are the terminal objects in some diagram in $\mathfrak{D}_i$ that otherwise takes values in $\mathcal{C}_i^n$. We require that $\mathcal{C} = \bigcup_n \mathcal{C}_i^n$.

Cartesian fibrations

Suppose $\mathcal{C}$ satisfies the required axioms, and let

$$\varpi : \mathcal{D} \to \mathcal{C}$$

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When $\mathcal{C} = \mathsf{FI}$ and ϖ is a right fibration, the Taylor coefficients of a $\mathcal{D}$-object carry the structure of a $\mathcal{D}$-object themselves, just as in the case of FI.

An example

Fix a manifold M and let $\mathbf{Braid}_M \stackrel{\mathrm{def}}{=} \mathsf{FI} \downarrow M$ – i.e. the category in which objects are configurations of marked points in M and morphisms are braids.

$\mathbf{Braid}_M$ admits a functor calculus analogous to FI-calculus in which the Taylor coefficients carry the structure of a $\mathbf{Braid}_M$-object.

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